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NAVAL POSTGRADUATE SCHOOL

Monterey, California

DISCRETE TIME SERIES GENERATED
BY MIXTURES III:
AUTOREGRESSIVE PROCESSES (DAR(p))

by

P. A. Jacobs
and
P. A. W. Lewis

September 1978

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NAVAL POSTGRADUATE SCHOOL
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0. ABSTRACT

A scheme for obtaining a stationary sequence of discrete random variables which has p-th order Markov dependence and a specified marginal distribution is presented. This DAR(p) process, which is a particular p-th order Markov chain, has the physical and correlation structure of an autoregressive process of order p. The process and its transition matrix are determined by the specified marginal distribution and by several other parameters which, independently of the marginal distribution, determine the correlation structure. Correlational properties and initial conditions for stationarity of the process are studied. Asymptotic properties for the process are also obtained.

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1. INTRODUCTION

In Jacobs and Lewis (1978a) a scheme for obtaining a sequence of discrete random variables with first order Markov dependence and with a given marginal distribution is presented. The process is a Markov chain and has the physical and correlation structure of a first order autoregressive process. In this paper we extend this result and present a scheme for obtaining a stationary sequence of discrete random variables with p-th order Markov dependence and having a given marginal distribution. The process is a p-th order Markov chain and has the physical and correlation structure of a higher order autoregressive process, i.e. X_n depends explicitly on $X_{n-1},\dots,X_{n-p}$. As with the models in Jacobs and Lewis (1978a) the process is specified by the stationary marginal distribution of X_n and several other chosen parameters which, independently of the marginal distribution, determine the correlation structure.

This DAR(p) process is defined as follows. Let $\{Y_n\}$ be a sequence of independent identically distributed random variables taking values in E a discrete subset of the real line with distribution π. Let $\{V_n\}$ be a sequence of independent $\{0,1\}$-random variables with

$$P\{V_n = 1\} = 1 - P\{V_n = 0\} = \rho$$

for $0 \le \rho < 1$. Let $\{A_n\}$ be a sequence of independent random variables taking values in $\{1,2,\dots,p\}$ with $P\{A_n = i\} = \alpha_i \ge 0$, $i = 1,2,\dots,p$ where $\sum_{i=1}^{p} \alpha_i = 1$. Put

(1.1) $$X_n = V_n X_{n-A_n} + (1 - V_n) Y_n$$

for $n = 1,2,\ldots$. Then $\{X_n\}$ is called the DAR(p) process (discrete autoregressive process of order p). It will be shown in Section 3 that it is possible to obtain an initial distribution for $(X_{-p+1}, \ldots, X_1, X_0)$ which makes the sequence $\{X_n\}$ stationary with marginal distribution π.

In Section 2 we will study the correlational structure of the stationary DAR(p) process, obtaining Yule-Walker type equations and discussing their solution for the DAR(2) and DAR(3) process.

Section 3 discusses the asymptotic properties of the DAR(p) process; it is shown that the process is ϕ-mixing in the sense of Billingsley (1968) and consequently that, with the usual definitions, sample means, covariances and quantiles converge almost surely and are asymptotically normal.

The process as defined at (1.1) has the property that all of the serial correlations $r(k) = \text{corr}(X_{n+k}, X_n)$, $k = \pm 1, \pm 2, \ldots$, are greater than or equal to zero. This problem can be overcome quite simply if π is a symmetric distribution. The process then becomes a full analog to the standard linear AR(p) process (see e.g., Box and Jenkins, 1970). A discussion of the correlation properties of this extended DAR(p) process is given. When π is not symmetric it is necessary to consider schemes such as coupling two DAP(p) processes over a bivariate sequence $\{Y_n, Y'_n\}$, where the Y_n's and Y_n''s are negatively correlated. Since this leads into the question of multivariate DAR(p) processes, it will be explicated elsewhere.

2. CORRELATIONAL PROPERTIES

Let $\{X_n\}$ be a stationary DAR(p) process having marginal distribution π and parameters ρ, α_i, $i = 1,\ldots,p$, and let $m = E(Y_n) = E(X_n)$. From (1.1) it follows that the dependence in the DAR(p) process is due to the random selection procedure by which X_n chooses one of the Y_k; $k \leq n$. In other words it is possible to write X_n as a random index model over Y_k, $k \leq n$, although the indexing variables for each n are correlated. To derive the correlation structure it is easier to work with the autoregressive formulation (1.1). Note that with probability $\rho\alpha_j$, $X_{n+k} = X_{n+k-j}$ for $j = 1,\ldots,p$. This observation allows one to derive Yule-Walker equations for the serial correlations $r(k) = \mathrm{Corr}(X_{n+k}, X_n)$ quite simply. For example, if we center the X_n's to give $X'_n = X_n - m$, then multiplying (1.1) by X'_{n-1} and taking expectations

$$
\begin{aligned}
r(1) &= E(X'_n X'_{n-1}) \\
&= E(X'_{n-1} V_n X'_{n-A_n}) + E(X'_{n-1}(1-V_n)Y_n - m) \\
&= \rho\alpha_1 r(0) + \rho\alpha_2 r(1) + \cdots + \rho\alpha_p r(p-1).
\end{aligned}
$$

Continuing this way, the complete set of Yule-Walker equations is obtained:

$$
\begin{aligned}
&(2.1) \quad r(1) = \rho\alpha_1 r(0) + \rho\alpha_2 r(1) + \cdots + \rho\alpha_p r(p-1)\,, \\
&(2.2) \quad r(2) = \rho\alpha_1 r(1) + \rho\alpha_2 r(0) + \cdots + \rho\alpha_p r(p-2)\,, \\
&\qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\qquad \vdots \\
&(2.3) \quad r(p) = \rho\alpha_1 r(p-1) + \rho\alpha_2 r(p-2) + \cdots + \rho\alpha_p r(0)\,,
\end{aligned}
$$

and, for $k \geq 1$,

$$(2.4) \quad r(p+k) = \rho\alpha_1 r(p+k-1) + \rho\alpha_2 r(p+k-2) + \cdots + \rho\alpha_p r(k) \quad .$$

Clearly, when $\rho = 0$ all $r(k)$'s, $k = 1,2,\ldots$ are zero, so that one is interested in solving the equations for $0 < \rho < 1$.

To see that the serial correlations are all positive or zero, let $q(i)$ be the probability that X_n and X_{n+i} choose the same random variable Y_k, where, because of the backward definition of the autoregression, $k \leq n$. Then $q(i)$ also satisfies equations (2.1)-(2.4). Hence $r(i) = q(i)$, $i = 1,2,\ldots$ and the correlations for the DAR(p) process are always nonnegative.

If we define a generating function

$$\beta(z) = \sum_{j=1}^{\infty} z^j\, r(j) \quad \text{for} \quad 0 \leq z \;,$$

then from (2.1) - (2.4) we have that

(2.5) $\beta(z)$

$$= \{z[\rho\alpha_1 r(0) +\cdots+ \rho\alpha_p r(p-1)] + z^2[\rho\alpha_2 r(0) +\cdots+ \rho\alpha_p r(p-2)] + z^p[\rho\alpha_p r(0)]\} \{1 - z\rho\alpha_1 - z^2\rho\alpha_2 - \cdots -z^p\rho\alpha_p\}^{-1}.$$

The roots of the characteristic equation

$$1 - z\rho\alpha_1 - z^2\rho\alpha_2 -\cdots- z^p\rho\alpha_p = 0 \tag{2.6}$$

can be used to expand $\beta(z)$ by partial fractions to obtain expressions for $r(k)$. The roots can also be used to characterize the behavior of the autoregressive function $\{r(k)\}$. This will be illustrated for the DAR(2) and DAR(3) processes.

First we note that since ρ and the α_i's are greater than or equal to zero, the left-hand side of (2.6) equals 1 at $z = 0$, $(1-\rho) > 0$ at $z = 1$, and since it is monotone in z for $z \geq 0$, there is one and only one positive root for a value of $z > 1$. All other roots are negative or imaginary. (For $p = 1$, the root is $z_1 = 1/\rho$.)

Another consequence of (2.6) concerns the quantity c defined in Jacobs and Lewis (1978b). We have

$$\lim_{m \to \infty} m \operatorname{var}\left(\frac{X_1 +\cdots+ X_m}{m}\right) = \operatorname{var}(X_1)c,$$

where

$$c = 1 + 2 \sum_{j=1}^{\infty} r(j) .$$

From (2.5)

$$(2.7) \qquad c = 1 + 2\beta(1)$$

$$= 1 + 2\rho\{r(0) + (1-\alpha_1)\, r(1) + (1-\alpha_1-\alpha_2)\, r(2)$$

$$+\cdots+ (1-\alpha_1 - \cdots - \alpha_{p-1})\, r(p-1)\}/(1-\rho)$$

$$\geq 1 + \frac{2\rho}{1-\rho}\ .$$

The inequality follows because all of the serial correlations $r(k)$ are nonnegative.

Example 2.1. The DAR(2) process.

In the definition (1.1) let $P\{A_n=1\} = 1 - P\{A_n=2\} = \alpha$ for $0 \leq \alpha \leq 1$. If $\alpha = 1$ the DAR(2) process reduces to the DAR(1) process which was studied in Jacobs and Lewis (1978 a,b). The Yule-Walker equations are easily solved to give

$$(2.7) \qquad r(1) = \rho\alpha[1 - \rho(1-\alpha)]^{-1}$$

and

$$(2.8) \qquad r(2) = \{\rho(1-\rho) + 2\rho\alpha[\rho-1]\}\{1 - \rho(1-\alpha)\}^{-1}$$

$$= r(1)[(1-\rho)\alpha^{-1} + 2\rho - 1]\ .$$

If $\alpha = 1$, $r(1) = \rho$, $r(2) = \rho^2$, .. as it should for a first order Markov chain; if $\alpha = 0$, then $r(1) = 0$, $r(2) = \rho$, $r(3) = 0$, $r(4) = \rho^2$, ... To obtain possible values of $r(1)$ and $r(2)$ note that from (2.7)

$$r(1)\ (1-r(1))^{-1} = \alpha\rho(1-\rho)^{-1} .$$

Thus for fixed $r(1)$, ρ must be greater than or equal to $r(1)$ since $0 \leq \alpha \leq 1$. Further, for fixed $r(1)$ and $\rho \geq r(1)$

(2.9) $$\alpha = r(1)\ [1 - r(1)]^{-1}\ (1-\rho)\ \rho^{-1} .$$

Suppose $r(1)$ is fixed and $\rho \geq r(1)$; then substituting (2.9) into (2.8) yields

$$r(2) = -r(1) + \rho[1 + r(1)] \geq r(1)^2 .$$

It now follows that the admissible region for $r(1)$ and $r(2)$ is $1 > r(2) \geq r(1)^2$.

Note that

(2.10) $$\frac{r(1) + r(2)}{1 + r(1)} = \rho$$

and

$$\rho\alpha = \frac{r(1)\ [1 + r(2)]}{1 - r(1)^2} .$$

Hence, it is possible to obtain estimates for ρ and α from estimates of the first two serial correlations.

To study the autoregressive function note that the characteristic equation (2.6) becomes

$$1 - z\rho\alpha - z^2\rho(1-\alpha) = 0$$

which has two real roots, for $0 \leq \alpha < 1$,

$$z_{1,2} = \{\rho\alpha \pm a\}\{-2\rho(1-\alpha)\}^{-1},$$

where

$$a = [(\rho\alpha)^2 + 4\rho(1-\alpha)]^{1/2}.$$

Since $a \geq [(\rho\alpha)^2]^{1/2} = \rho\alpha$, one root is positive, and the other is negative. Also the negative root is dominant in absolute value. Thus, the autocorrelation function for the DAR(2) process is non-negative and is a mixture of the larger inverse of the positive root decaying geometrically and the smaller (in absolute value) inverse of the negative root decaying in absolute value geometrically but oscillating in sign. Specifically,

$$r(k) = b_1 z_1^{-k} + b_2 z_2^{-k} \tag{2.12}$$

for b_1 and b_2 given as

$$b_1 = z_1^{-1}[1 - z_2^{-2}]\,\{(z_1^{-1} - z_2^{-1})(1 + z_1^{-2}z_2^{-2})\}^{-1}$$

$$b_2 = z_2^{-1}(1 - z_1^{-2})\,\{(z_1^{-1} - z_2^{-1})(1 + z_1^{-1}z_2^{-2})\}^{-1}.$$

Eventually the larger inverse root z_2^{-1} dominates in $r(k)$ and the decay is geometric.

Example 2.2. The DAR(3) process.

In definition (1.1) let

$$P\{A_n=1\} = \alpha_1, \quad P\{A_n=2\} = \alpha_2, \quad P\{A_n=3\} = 1 - \alpha_1 - \alpha_2$$

for $0 \leq \alpha_1$, $0 \leq \alpha_2$, $0 \leq \alpha_1 + \alpha_2 \leq 1$. If $\alpha_1 + \alpha_2 = 1$ then the process reduces to the DAR(2) process. The Yule-Walker equations become

$$(2.13) \quad r(1) = \rho\alpha_1 + \rho\alpha_2 r(1) + \rho(1 - \alpha_1 - \alpha_2)r(2);$$

$$(2.14) \quad r(2) = \rho\alpha_1 r(1) + \rho\alpha_2 + \rho(1 - \alpha_1 - \alpha_2)r(1);$$

$$(2.15) \quad r(3) = \rho\alpha_1 r(2) + \rho\alpha_2 r(1) + \rho(1 - \alpha_1 - \alpha_2);$$

$$(2.16) \quad r(k) = \rho\alpha_1 r(k-2) + \rho\alpha_2 r(k-2) + \rho(1 - \alpha_1 - \alpha_2)r(k-3)$$

for $k \geq 3$. Thus from (2.13) and (2.14)

$$r(1) = [\rho\alpha_1 + \rho^2(1 - \alpha_1 - \alpha_2)\alpha_2][1 - \rho\alpha_2 - \rho^2(1 - \alpha_1 - \alpha_2)(1 - \alpha_2)]^{-1};$$

$$r(2) = \rho\alpha_2 + \rho(1 - \alpha_2)r(1);$$

and

$$r(3) = \rho^2\alpha_1\alpha_2 + \rho(1 - \alpha_1 - \alpha_2) + \{\rho^2\alpha_1(1 - \alpha_2) + \rho\alpha_2\}r(1) .$$

Since equations (2.13)-(2.16) are linear in $\alpha_1' = \rho\alpha_1$, $\alpha_2' = \rho\alpha_2$, and ρ with coefficients involving $r(1)$, $r(2)$ and $r(3)$, they can be solved for α_1', α_2' and ρ, and thus for α_1, α_2 and ρ. To facilitate this note that (2.13)-(2.15) can be rewritten as

$$(2.13') \qquad r(1) = \alpha_1'\{1 - r(2)\} + \alpha_2'[r(1) - r(2)] + \rho r(2);$$

$$(2.14') \qquad r(2) = \alpha_2'[1 - r(1)] + \rho r(1);$$

$$(2.15') \qquad r(3) = \alpha_1'[r(2) - 1] + \alpha_2'[r(1) - 1] + \rho\ .$$

Hence estimates for ρ, α_1, and α_2 can be obtained from the usual estimates of the serial correlations $r(1)$, $r(2)$ and $r(3)$. Problems will arise in solving the equations (2.13')-(2.15') if ρ is close to zero, since when $\rho = 0$ all the quantities in the equation are identically zero.

The characteristic equation (2.6) becomes

$$(2.17) \qquad 0 = 1 - \rho\alpha_1 z - \rho\alpha_2 z^2 - \rho(1 - \alpha_1 - \alpha_2)z^3\ .$$

Again we require that $\alpha_1 + \alpha_2 < 1$, so that the process does not reduce to the DAR(2) case. To find the roots of (2.17), let

$$a = \frac{1}{3}\left[3\,\frac{\alpha_1}{1 - \alpha_1 - \alpha_2} - \left(\frac{\alpha_2}{1 - \alpha_1 - \alpha_2}\right)^2\right],$$

$$b = \frac{1}{27}\left\{2\left(\frac{\alpha_2}{1 - \alpha_1 - \alpha_2}\right)^2 - 9\,\frac{\alpha_1\alpha_2}{(1 - \alpha_1 - \alpha_2)^2} - 27\,\frac{1}{\rho(1 - \alpha_1 - \alpha_2)}\right\}$$

and

$$d = \frac{b^2}{4} + \frac{a^3}{27} .$$

The value of d can be either positive or negative. If $d > 0$, equation (2.17) has one real positive root and two imaginary roots and the autocorrelation function will have a "damped sine wave" appearance, but the function will eventually decrease geometrically. If $d < 0$, then all the roots of the cubic are real and unequal, one being positive and greater than one, the other two being negative. Hence the autocorrelation function is a mixture of a geometrically decreasing term, and two terms which decrease geometrically in absolute value but oscillate in sign. Table 1 gives values for the first nine correlations for a DAR(3) process with $\rho = 0.9$ for various values of α_1 and α_2.

Example 2.3. The DAR(p) process.

The Yule-Walker equations (2.1)-(2.4) can be reparametrized with $\phi_1 = \rho\alpha_1$, $\phi_2 = \rho\alpha_2, \ldots, \phi_p = \rho\alpha_p$, the parameters all lying between 0 and 1, with the constraint that

(2.18) $$0 \leq \phi_1 + \phi_2 + \cdots + \phi_p < 1 .$$

Then it can be seen that the Yule-Walker equations are in exactly the same form as those given for the usual AR(p) process in Box and Jenkins (1970, p. 55). The parameter space for the DAR(p)

TABLE 1

VALUES FOR THE FIRST NINE CORRELATIONS FOR A DAR(3) PROCESS

WITH $\rho = 0.9$ FOR VARIOUS VALUES OF α_1 AND α_2.

(Recall that $\alpha_1 + \alpha_2 < 1$.)

α_1	α_2	r(1)	r(2)	r(3)	r(4)	r(5)	r(6)	r(7)	r(8)	r(9)
0	0	0	0	.9	0	0	.81	0	0	.729
0	0.5	.58	.71	.71	.58	.64	.58	.55	.55	.51
0	0.9	.40	.85	.41	.72	.41	.62	.40	.54	.38
0.05	0.9	.44	.85	.44	.73	.43	.63	.41	.55	.38
0.1	0.8	.59	.83	.59	.70	.56	.61	.52	.53	.48
0.3	0.6	.75	.81	.71	.70	.64	.61	.58	.54	.51
0.45	0.45	.80	.80	.74	.70	.65	.61	.58	.54	.51
0.5	0	.76	.68	.76	.68	.61	.62	.58	.54	.52
0.6	0.3	.84	.80	.75	.69	.65	.60	.56	.53	.49
0.8	0.1	.87	.79	.74	.68	.63	.58	.54	.50	.46
0.9	0	.88	.79	.73	.67	.62	.57	.52	.48	.44

process is more restricted than that for $\phi_1, \ldots, \phi_p$ in the AR(p) case, but the solutions are the same. The parameter ρ does not appear explicitly in this parametrization; however the parametrization is useful in the next section and when negative correlation is considered.

3. ASYMPTOTIC PROPERTIES

3.1. Preliminaries: Aperiodicity and the Doeblin Condition

Let $\{X_n\}$ be a real-valued DAR(p) process with discrete state space E having parameters ρ and $\{\alpha_i;\ i = 1,2,\ldots,p\}$ and with distribution π for the error sequence Y_n. From the definition (1.1) it follows that $Z_n = (X_n, X_{n+1}, \ldots, X_{n+p-1})$ is a Markov chain having state space F which is the pth-order product space of E with itself. Let P denote the transition matrix of $\{Z_n\}$. Then for $\underset{\sim}{i}, \underset{\sim}{k} \in F$, $\underset{\sim}{k} = (k_1,\ldots,k_p)$, $\underset{\sim}{i} = (i_1,\ldots, i_p)$

$$(3.1) \qquad P(\underset{\sim}{i},\underset{\sim}{k}) \geq (1-\rho)^p\, \pi(k_1)\cdots \pi(k_p)\ .$$

For C a subset of F let

$$\mu(C) = \sum_{\underset{\sim}{k}\in C} \pi(k_1) \cdots \pi(k_p) = \sum_{k\in C} \mu(k)\ .$$

The asymptotic theory for DAR(p) processes is based on the following result.

PROPOSITION (3.1). The Markov chain $\{Z_n\}$ is aperiodic and satisfies the Doeblin hypothesis D as generalized by Doob [p. 192, [1953]].

PROOF. Fix a subset C of F such that

$$\delta = \inf_{\underset{\sim}{k} \in C} \mu(\underset{\sim}{k}) > 0 .$$

Let A be a subset of F with $\mu(A) \leq \mu(C)/2$. Then

$$\begin{aligned}
& P(\underset{\sim}{i}, A^c \cap C) \\
& \quad = \sum_{\underset{\sim}{k} \in A^c \cap C} \frac{P(\underset{\sim}{i},\underset{\sim}{k})}{\mu(\underset{\sim}{k})} \mu(\underset{\sim}{k}) \\
& \quad \geq \sum_{\underset{\sim}{k} \in A^c \cap C} (1-\rho)^p \mu(\underset{\sim}{k}) \\
& \quad = (1-\rho)^p \mu(A^c \cap C) \\
& \quad \geq (1-\rho)^p \mu(C)/2 \\
& \quad \geq (1-\rho)^p \delta/2 ,
\end{aligned}$$

since $\mu(A) \leq \mu(C)/2$. Hence,

$$\begin{aligned}
(3.2) \qquad P(\underset{\sim}{i},A) &= 1 - P(\underset{\sim}{i}, A^c) \\
&\leq 1 - P(\underset{\sim}{i}, A^c \cap C) \\
&\leq 1 - \frac{1}{2} (1 - \rho)^2 \delta .
\end{aligned}$$

The result now follows.

We will use proposition (3.1) to show existence of a unique limiting distribution for $\{Z_n\}$, and to show that the stationary DAR(p) process is ϕ-mixing.

3.2. Stationarity Conditions

It follows from Proposition (3.1) that there exists a unique limiting distribution ν for $\{Z_n\}$ and constants $\gamma \geq 0$ and β, $0 \leq \beta < 1$ such that

$$|P^n(\nu,A) - \nu(A)| \leq \gamma\beta^n , \qquad n = 1,2,... \tag{3.3}$$

for all subsets A of F (cf. Doob [1953, p. 221]). A stationary DAR(p) process is obtained by using ν as the initial distribution for $(X_{-p+1}, X_{-p+2}, \ldots, X_0)$. To see that ν has marginal distribution π, let $\{\gamma_n(j), j \in E\}$ denote the marginal distribution of the first component of Z_n and put $\gamma(j) = \lim_{n\to\infty} \gamma_n(j)$. Note that

$$\lim_{n\to\infty} \gamma_n(j) = \sum_{i=1}^{p} \rho\alpha_i \lim_{n\to\infty} \gamma_{n-i}(j) + (1-\rho)\,\pi(j) .$$

Hence, $\gamma(j) = \rho\gamma(j) + (1-\rho)\,\pi(j)$ and the marginal distribution of the stationary DAR(p) process is π as asserted in Section 1.

Example 3.1. The DAR(2) process.

Let $\{X_n\}$ be a DAR(2) process with the same parametrization as in Example 2.1. If

$$\gamma(s_1,s_2) = \lim_{n\to\infty} E[\exp\{-s_1X_{n+1} - s_2X_{n+2}\}] , \tag{3.4}$$

and

$$\beta(s_1) = \lim_{n\to\infty} E[\exp\{-s_1X_n\}]$$

then $\gamma(s_1,s_2)$ satisfies the equation

$$(3.5) \quad \gamma(s_1,s_2) = (1-\rho)\ \beta(s_1)\ \beta(s_2) + \rho\alpha\beta(s_1+s_2) + \rho(1-\alpha)\ \gamma(s_1,s_2)\ .$$

Solving (3.5) we get

$$\gamma(s_1,s_2) = \gamma(s_2,s_1) = (1 - r(1))\ \beta(s_1)\ \beta(s_2) + r(1)\ \beta(s_1+s_2)$$

and inverting the transform results in

$$(3.6) \quad \nu(i,j) = \lim_{n\to\infty} P\{X_{n+1}=i, X_{n+2}=j\} = \begin{cases} (1-r(1))\ \pi(i)\ \pi(j), & i\neq j, \\ r(1)\ \pi(j) + (1-r(1))\ \pi(j)^2, & i=j. \end{cases}$$

where $r(1) = \alpha\rho\{1 - \rho(1-\alpha)\}^{-1} = \mathrm{corr}(X_n, X_{n+1})$ in the stationary process. Note that $\sum_{j\in E} \nu(i,j) = \sum_{j\in E} \nu(j,i) = \pi(i)$, as expected.

In simulating a random variable with distribution ν it is simpler to use the conditional distribution

$$\nu(j|i) = \lim_{n\to\infty} P\{X_{n+2}=j|X_{n+1}=i\} = \begin{cases} \{1 - r(1)\}\ \pi(j)\ , & j \neq i, \\ r(1) + (1-r(1))\ \pi(i), & j = i. \end{cases}$$

Note that we can now generate a stationary sequence of random variables X_n, $n = 1,2,\ldots$ with any marginal distribution π and second-order autoregressive (AR(2)) correlation structure. From (3.6) or the conditional distribution it is not hard to show that, with $m = E(Y_n) = E(X_n)$,

$$(3.7) \qquad E[X_n | X_{n-1} = x_{n-1}] = \{1 - r(1)\}m + r(1)x_{n-1},$$

$$(3.8) \qquad E[X_n | X_{n+1} = x_{n+1}] = \{1 - r(1)\}m + r(1)x_{n+1},$$

$$(3.9) \qquad \mathrm{Var}[X_n | X_{n-1} = x_{n-1}] = \{1 - r(1)\}\, E(X_n^2) + r(1)\{1 - r(1)\}x_{n-1}^2 - \{1 - r(1)\}^2 m^2 - 2r(1)(1 - r(1))mx_{n-1}.$$

Directly from (1.1) we have, for example, that

$$(3.10) \quad E[X_n | X_{n-1} = x_{n-1};\ X_{n-2} = x_{n-2}] = \rho\alpha x_{n-1} + \rho(1-\alpha)x_{n-2} + (1-\rho)m;$$

$$(3.11) \quad \mathrm{Var}[X_n | X_{n-1} = x_{n-1};\ X_{n-2} = x_{n-2}] = (1-\rho)\ \mathrm{Var}(Y_n) + \rho\alpha(1-\rho\alpha)x_{n-1}^2 + \rho(1-\alpha)[1-\rho(1-\alpha)]x_n^2 + (1-\rho)\ \rho E[Y_n^2] + 2\rho^2\alpha(1-\alpha)x_{n-1}x_n + 2\rho(1-\rho)m[\alpha x_{n-1} + (1-\alpha)x_n]\ .$$

The regressions (3.7) and (3.8) might suggest that the DAR(2) process is time-reversible, but this is not so. To see this, note that

$$\gamma(s_1,s_2,s_3) = \lim_{n\to\infty} E[\exp\{-s_1X_{n+1} - s_2X_{n+2} - s_3X_{n+3}\}]$$

$$= \rho\alpha\gamma(s_1,s_2+s_3) + \rho(1-\alpha)\ \gamma(s_1+s_3,s_2)$$

$$+ (1-\rho)\ \beta(s_3)\ \gamma(s_1,s_2) \quad .$$

Since $\gamma(s_1,s_2,s_3) \neq \gamma(s_3,s_2,s_1)$ it follows that the stationary DAR(2) process is not time reversible.

Example 3.2. The DAR(3) process.

Let $\{X_n\}$ be a DAR(3) process with the same parametrization as in Example (2.2). As before let

$$\gamma(s_1,s_2) = \lim_{n\to\infty} E[\exp\{-s_1X_{n+1} - s_2X_{n+2}\}] \ ,$$

and let

$$\beta(s_1) = \lim_{n\to\infty} E[\exp\{-s_1X_n\}];$$

$$\delta(s_1,s_2) = \lim_{n\to\infty} E[\exp\{-s_1X_{n-1} - s_2X_{n+1}\}];$$

and

$$\Gamma(s_1,s_2,s_3) = \lim_{n\to\infty} E[\exp\{-\{s_1X_{n+1} - s_2X_{n+2} - s_3X_{n+3}\}] \ .$$

Put $\phi_1 = \rho\alpha_1$, $\phi_2 = \rho\alpha_2$, and $\phi_3 = \rho(1 - \alpha_1 - \alpha_2)$. Then, from the definition (1.1), and accounting for the possibility of X_{n+3} taking values X_{n+2}, X_{n+1}, X_n or Y_{n+3}, we have

$$(3.12) \quad \Gamma(s_1,s_2,s_3)$$

$$= (1-\rho)\ \beta(s_3)\ \gamma(s_1,s_2) + \phi_1\gamma(s_1,s_2+s_3)$$

$$+ \phi_2\gamma(s_1+s_3,s_2) + \phi_3\ \Gamma(s_3,s_1,s_2)\ .$$

Also

$$(3.13) \quad \gamma(s_1,s_2)$$

$$= (1-\rho)\ \beta(s_1)\ \beta(s_2) + \phi_1\beta(s_1+s_2) + \phi_2\gamma(s_2,s_1) + \phi_3\delta(s_1,s_2)$$

and

$$(3.14) \quad \delta(s_1,s_2)$$

$$= (1-\rho)\ \beta(s_1)\ \beta(s_2) + \phi_1\gamma(s_1,s_2) + \phi_2\gamma(s_1+s_2) + \phi_3\gamma(s_2,s_1).$$

Solving (3.13) and (3.14) simultaneously for $\gamma(s_1,s_2)$ and $\delta(s_1,s_2)$, it follows that

$$\gamma(s_1,s_2) = (1 - r(1))\ \beta(s_1)\ \beta(s_2) + r(1)\ \beta(s_1 + s_2)\ ,$$

where $r(1) = \text{corr}(X_n,X_{n+1})$ is given in Example (2.2). Hence, inverting the transform,

$$\text{(3.15)} \qquad \nu(i,j) = \begin{cases} (1-r(1))\ \pi(i)\ \pi(j)\ , & \text{if } i \neq j, \\ r(1)\ \pi(j)\ , & \text{if } i = j. \end{cases}$$

The expression (3.12) can now be used to obtain an expression for $\Gamma(s_3,s_1,s_2)$, and substituting back into (3.12) we get

$$\begin{aligned}
\text{(3.16)} \quad & \Gamma(s_1,s_2,s_3) \\
&= \{1-\phi_3^3\}^{-1} \{(1-\rho)[\beta(s_3)\gamma(s_1,s_2) + \phi_3\beta(s_2)\gamma(s_3,s_1) \\
&\quad + \phi_3^2\beta(s_1)\gamma(s_2,s_3)] + \phi_1[\gamma(s_1,s_2+s_3) + \phi_3\gamma(s_3,s_1+s_2) \\
&\quad + \phi_3^2\gamma(s_2,s_3+s_1)] + \phi_2[\gamma(s_1+s_3,s_2) + \phi_3\gamma(s_3+s_2,s_1) \\
&\quad + \phi_3^2\gamma(s_2+s_1,s_3)]\} \quad .
\end{aligned}$$

Note that $\Gamma(s_1,s_2,s_3) \neq \Gamma(s_3,s_2,s_1)$ in general and hence the stationary DAR(3) process is not time reversible.

3.3. Mixing Results

The following result follows directly from (3.3).

PROPOSITION 3.2. The stationary DAR(p) process is ϕ-mixing in the sense of Billingsley [1968] with $\phi(k) = \gamma\beta^k$.

This last result implies that all of the asymptotic results in Jacobs and Lewis [1978b] are true for the DAR(p) process modulo constants. In particular the usual estimates of covariances, percentiles and quantiles converge almost surely to the true covariances, percentiles and quantiles and the estimates are asymptotically normal. Further, a chi-square goodness of fit test can be obtained for the marginal distribution of the stationary DAR(p) process. For further details see Jacobs and Lewis [1978b].

4. EXTENSIONS

One possible drawback of the DAR(p) process as defined in (1.1) is that all of the correlations are positive. We will now discuss a variation of the DAR(p) process which allows negative correlation for the case in which the marginal distribution π is symmetric about zero. The case of nonsymmetric π is more difficult and will be considered elsewhere.

Let $\{V_n\}$, $\{A_n\}$, and $\{Y_n\}$ be as in Section 1 and suppose that π is symmetric about zero. Let

$$(4.1) \qquad X_n = V_n a_{A_n} X_{n-A_n} + (1 - V_n) Y_n ,$$

where $\{a_k;\ k = 1,\dots,p\}$ is a fixed sequence of numbers that are either -1 or 1.

Put $Z_n = (X_n, X_{n+1}, \dots, X_{n+p-1})$. Then $\{Z_n\}$ is a Markov chain with a transition matrix P which satisfies (3.1). Thus Propositions (3.1) and (3.2) still hold, as well as (3.3).

Let $\{X_n\}$ be a stationary process defined by the modified DAR(p) scheme of (4.1) and put $r(k) = \mathrm{corr}(X_n, X_{n+k})$. The Yule-Walker equations for this process are

$$(4.2)\qquad r(1) = \phi_1 + \phi_2 r(1) + \cdots + \phi_p r(p-1);$$

$$(4.3)\qquad r(2) = \phi_1 r(1) + \phi_2 + \cdots + \phi_p r(p-2);$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$(4.4)\qquad r(p) = \phi_1 r(p-1) + \phi_2 r(p-2) + \cdots + \phi_p r(0);$$

$$(4.5)\qquad r(p+k) = \phi_1 r(p+k-1) + \phi_2 r(p+k-2) + \cdots + \phi_p r(k)\ ;$$

for $k \geq 1$ where $\phi_k = a_k \rho \alpha_k$, $k = 1,\ldots,p$.

Equations (4.2)-(4.4) have the same form as the Yule-Walker equation on page 55 of Box and Jenkins (1970) but the set of possible values of ϕ_i, $i = 1,\ldots,p$ is more restrictive. In particular we always have that

$$(4.6)\qquad |\phi_1| + |\phi_2| + \cdots + |\phi_p| < 1$$

<u>Example 4.1</u>. The modified DAR(1) process.

Let $A_n \equiv 1$ and $a_1 = -1$. Then

$$X_{n+1} = -V_{n+1} X_n + (1 - V_{n+1}) Y_{n+1}$$

and $r(k) = (-1)^k \rho^k$. If $a_1 = 1$ the process is the same as that in Section 1. When $a_1 = -1$ the modified DAR(1) process is Markovian with transition matrix

$$P(i,j) = P\{X_{n+1} = j \mid X_n = i\} = \begin{cases} (1-\rho)\ \pi(j), & -i \neq j\ , \\ \rho + (1-\rho)\pi(j), & -i = j\ . \end{cases}$$

Example 4.2. The modified DAR(2) process.

Let $P\{A_n = 1\} = 1 - P\{A_n = 2\} = \alpha$, $0 \leq \alpha < 1$. Solving equations (4.2)-(4.4) we obtain

$$r(1) = a_1\rho\alpha[1 - a_2\rho(1-\alpha)]^{-1},$$

$$r(2) = \rho[a_2(1-\alpha) - \rho(1 - 2\alpha)][1 - a_2\rho(1-\alpha)]^{-1}\ ,$$

and

$$r(3) = \rho^2\alpha[2a_1a_2(1-\alpha) - a_1\rho(1 - 2\alpha)]\ [1 - a_2\rho(1-\alpha)]^{-1}\ .$$

Note that if $a_1 = 1$ (respectively $a_1 = -1$), the $r(1) > 0$, (respectively $r(1) < 0$) and $r(2)$ can be either positive or negative. The analogue of the characteristic equation (2.6) is

$$1 - a_1\rho\alpha z - a_2\rho(1-\alpha)z^2 = 0\ . \tag{4.6}$$

If $a_2 = 1$, then (4.6) has two real roots; if $a_2 = -1$, then (4.6) may have imaginary roots. Thus the autocorrelation function can either have a "damped sine wave" appearance or be a mixture of terms which decrease geometrically in absolute value but may oscillate in sign.

For the modified DAR(2) process the range of values which the parameters ϕ_1 and ϕ_2 can take is not as broad as for the AR2 case, for which we have (Box and Jenkins [1970, p. 58])

$$\phi_2 + \phi_1 < 1,$$
$$\phi_2 - \phi_2 < 1,$$
$$-1 < \phi_2 < 1,$$

which allows ϕ_1 to range between -2 and +2. By contrast for the modified DAR(2) (for symmetric distributions) we have

$$|\phi_2| + |\phi_1| < 1,$$
$$-1 < \phi_1 < 1,$$
$$-1 < \phi_2 < 1.$$

Thus we have a diamond of possible values in the (ϕ_1, ϕ_2) plane, with apexes (0,1), (1,0), (0,-1), (-1,0). With this restriction the behavior of the serial correlations for the modified DAR(2) process is the same as the behavior for the AR(2) process which is pictured in the figure in Box and Jenkins (1970).

A limiting stationary distribution can be derived for this modified DAR(2) process, as was done for the ordinary DAR(2) process in Example 3.1.

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